AFRL-SR-BL-TR-98-

0399

# REPORT DOCUMENTATION PAGE



| 1. AGENCY USE ONLY *(Leave blank)* | 2. REPORT DATE | 3. REPORT TYPE AND DATES COVERED |
|---|---|---|
| | MARCH 27, 1998 | FINAL REPORT: 4/1/97-12/31/97 |

4. TITLE AND SUBTITLE
Improved techniques for modeling and controlling nonlinear systems with few degrees of freedom

5. FUNDING NUMBERS
AFOSR F49620-97-1-0158

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8. PERFORMING ORGANIZATION REPORT NUMBER
333261

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM, Suite B115
110 Duncan Ave.
Bolling AFB, DC 20332-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

New nonlinear signal processing and modeling techniques were examined. The key issues which formed the focus of the project: were 1] The design of coupling terms for synchronizing the model with driving signals from the system of interest. This is the first analytical result of its kind, and gives sufficient conditions for guaranteeing that the model will synchronize. 2] The development of symbolic time series methods for estimatiion of correlation timescales of complex signals and the detection of weak periodicities. 3] A theory of anomalous scaling behavior in nonnomal systems which are subcritical (i.e. linearly stable but unstable to finite amplitude perturbations). Both noise-free and noise-driven cases were treated.

19980410 053

14. SUBJECT TERMS
Signal, modeling, control, nonlinear, synchronization, non-normal, bifurcation

15. NUMBER OF PAGES
12(incl. cover sheet)

16. PRICE CODE

| 17. SECURITY CLASSIFICATION OF REPORT | 18. SECURITY CLASSIFICATION OF THIS PAGE | 19. SECURITY CLASSIFICATION OF ABSTRACT | 20. LIMITATION OF ABSTRACT |
|---|---|---|---|
| UNCLASSIFIED | UNCLASSIFIED | UNCLASSIFIED | UL |

NSN 7540-01-280-5500

Computer Generated

STANDARD FORM 298 (Rev 2-89)
Prescribed by ANSI Std 239-18
298-102

# Improved Techniques for Modeling and Controlling Nonlinear Systems with Few Degrees of Freedom

**Final Report for Grant No. AFOSR F49620-97-1-0158**
from
The College of William and Mary

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## Executive Summary

### Statement of objectives

The research carried out under this grant concerned the development of improved diagnostic, modeling and control techniques. The techniques are built upon observational data and are tailored to the problem at hand. The models used are global discrete time mappings, ordinary and partial differential equations.

A major effort during the program was an examination of observability and detectability for nonlinear dynamical systems. The focus was on developing methods for designing coupling that guarantee observability or detectability between plant and model. Our results are analytical and can be used to estimate the complete state of a nonlinear system from limited measurements.

A second major effort concerned the development of symbolic time series methods for estimation of correlation timescales and/or detection of periodicities in complex signals. This technique shows particular promise in high-noise situations, where it had previously been shown to be capable of robust parameter estimation.

A third major effort concerned the development of scaling laws for non-normal transitions. Such transitions might occur in shear flows or in aeroelastic applications and would be particularly difficult to control or avoid. A theoretical understanding of the scaling laws is necessary before one can distinguish such transitions from more standard cases.

### Methods employed

The methods used, and results from, the observability/detectability work are primarily analytical. They do not assume a particular form for the model plant or the coupling, and they can be used when the dynamics of the plant exhibits complex nonlinear motion. Although our published results employ linearization about particular measures of dynamics, subsequent work indicates that a full nonlinear analysis will yield the same observability/detectability criteria. The methods explicitly employ a measure which represents the dynamics. This sensitivity to the measure is inherent to all observer problems and can not be avoided. However, our methods and results have their simplest analytic form on the most unstable measures. Thus, the most important measures are the easiest to evaluate. The methods yield rigorous observability/detectability results. Because they are based on norms these results are sufficient, but not necessary.

### Significance of the proposed activity

The proposed research will significantly enhance the ability to detect, model and control the dynamics of low-dimensional nonlinear systems using observed time series data. The observers and detectors that arise from the methods can estimate the full state of a nonlinear plant even when it exhibits complicated nonlinear motion. Thus, full state control algorithms can be used even when the measurements consist of scalar time series. The approach used comprehensive and can be implemented using experimentally obtained data from a diverse group of sources.

Contents

## C. RESEARCH OBJECTIVES

The goal of this research program is the development of new robust algorithms for diagnosing, modeling, and controlling nonlinear systems that have only a few degrees of freedom. (In this context the number of *degrees of freedom* is equal to the number of independent variables needed to accurately model the system which generated the data.) While systems with a few degrees of freedom have been the focus of the research, an important result is that some of the techniques are also applicable to distributed parameter systems exhibiting complicated spatio-temporal dynamics. Such systems are typically modeled as partial differential equations and, in principle, have an infinite number of degrees of freedom.

Time-series data, in conjunction with whatever *a priori* information is known about the system, are used as input to the modeling process. These methods are robust to uncertainties in the models, and to the presence of noise [1–5]. These techniques have been successfully applied to data from chemical reactions, electronic circuits, and mechanical systems [1]. Other researchers are also beginning to apply these methods to experimental data [6–8].

## D. BACKGROUND

### 1. Observers, detectors, and synchronization

Observers and detectors are often used to estimate the full state of a system given time series measurements. In most examples the plants and observers are linear. Here, we permit both to be nonlinear, and we assume the uncontrolled dynamics of the plant may be complicated. The approach couples the dynamics of a nonlinear plant to an observer via drive–response coupling

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{F}(\mathbf{x};t)\\ \frac{d\mathbf{y}}{dt} &= \mathbf{F}(\mathbf{y};t) - \mathbf{E}\cdot(\mathbf{y}-\mathbf{x}),\end{aligned} \tag{1}$$

where $\mathbf{x} \in I\!R^d$ represents the dynamics of the plant and $\mathbf{y} \in I\!R^d$ represents the dynamics of the observer. Here we ignore modeling errors, having examined these issues in a previous paper [9] (see also new results [10]). The coupling, $\mathbf{E}$, is a vector function of its argument with $\mathbf{E}(\mathbf{0}) = \mathbf{0}$. For this work evaluating $\mathbf{E}$ does not necessarily require all components of $\mathbf{x}$. Because the models are deterministic we know that if $\mathbf{y} = \mathbf{x}$ at any $t = t_*$ then $\mathbf{y} = \mathbf{x}$ for all $t > t_*$. Thus, if one can determine $\mathbf{E}$ such that $\lim_{t\to\infty} |\mathbf{y} - \mathbf{x}| = 0$ then one can estimate, $\mathbf{x}$, the complete state of the plant by $\hat{\mathbf{x}} = \mathbf{y}$.

Within the dynamical systems community this area of research is called synchronization. The dynamics of Eqs. (1) is said to be synchronized if $\mathbf{x} = \mathbf{y}$. Within this community one says that $\mathbf{x} = \mathbf{y}$ is an invariant manifold for the full dynamics, and one is interested in the existence and stability of dynamics on this manifold.

### 2. Symbolic time series analysis

Recent work by us [4,5,11] has shown that it is possible to convert an analog signal into a binary string while still retaining much of the dynamical information present in the signal. It is possible to do reliable parameter estimation, even in the presence of noise, and to detect periodic behavior masked either by noise or a chaotic signal. The primary advantages of this approach over more traditional techniques are that it is robust to noise and runs fast in real time. Under the recently ended grant (for which this is the final report) we concentrated upon developing the means to extract correlation timescale estimates and detection of weak periodic signals. Future extensions include analysis of weakly non-stationary signals and spatio-temporal pattern analysis.

#### 3. Shear flow instabilities and non-normal dynamics

Consider the transition from laminar to turbulent flow in a fluid. A fluid has an infinite number of degrees of freedom, but there is ample experimental evidence to suggest that near transition only a few degrees of freedom play a role (see, e.g. [12,13] for some recent discussions). Physical intuition suggests that these degrees of freedom might be related to those modes which a linear analysis identifies as the most unstable just after transition, or those most weakly damped just prior to transition. Extending this to a fully nonlinear treatment is difficult, but is embodied in the center manifold theorem which asserts that at transition, assuming there are no *unstable* degrees of freedom, the long time behavior will be dominated by those degrees of freedom which are neutrally stable. In many cases this center manifold can be constructed explicitly by performing a Galerkin projection on the primitive equations leading to low-dimensional models [14,13]. Alternatively, low-dimensional models can be constructed directly from the time series. The work of [13] on transitional shear flows is an impressive example of work that combines elements of both approaches.

In some instances the flow goes unstable at a Reynolds' number which is far below that predicted by linear theory. A conjecture dating back to Orr [15] suggests that this is due to the fact that the linearized Navier-Stokes' operator is non-normal. In some case this leads to transient amplification of perturbations even though the system is stable at long times. This conjecture has been revived in recent years and led to a lively debate in the literature (see [16–20] for the 'pro' camp and [21] for the 'con'). In our work [22,23] we have examined the claim of [19] that non-normal systems exhibit anomalous scaling behavior. Such anomalous scaling behavior might provide an experimental signature which would distinguish these transitions from normal ones.

If such transitions occur in applications they would be extremely dangerous as they would be hard to control and extremely sensitive to perturbations.

## E. NEW RESULTS

#### 1. Synchronization and the observer problem

The major new result of this portion of the research program is a rigorous criteria which, if satisfied, guarantees that the invariant manifold given by $\mathbf{x} = \mathbf{y}$ is linearly stable. More importantly, the criteria can be used to *design* couplings, $\mathbf{E}$, which satisfy the criteria. The criteria only uses knowledge of the uncoupled dynamics, $\mathbf{F}$, and many of the important calculations can be performed analytically [3].

Given the following optimal decomposition [3]

$$\mathbf{A} = \langle \mathbf{DF} \rangle - \mathbf{DE}(\mathbf{0}), \tag{2}$$

(where $\mathbf{DF}$ and $\mathbf{DE}$ are Jacobians) the criteria for linear stability of the manifold $\mathbf{x} = \mathbf{y}$ is [3]

$$-\Re[\Lambda_1] > \left\langle \| \mathbf{P}^{-1} \left[ \mathbf{DF}(\mathbf{x}) - \langle \mathbf{DF} \rangle \right] \mathbf{P} \| \right\rangle, \tag{3}$$

where $\Lambda_1$ is the eigenvalue of $\mathbf{A}$ that has the largest real part, $\Re[\Lambda]$ is the real part of $\Lambda$, and $\mathbf{P}$ is a matrix composed of the eigenvectors of $\mathbf{A}$. Here $\langle \bullet \rangle$ denotes a time average along the driving trajectory, $\mathbf{x}$. The driving trajectory is one measure of the dynamics of the plant, and each possible driving trajectory may represents a different measure of the plant. Similar results by others lead us to conjecture that the criteria in Eqs. (3) can be obtained from a full nonlinear analysis [24].

Equations (2) and (3) represent definitions and criteria indicating when synchronous motion for a particular driving trajectory becomes unstable to small perturbations in directions transverse to the synchronization manifold. The criterion is rigorous and sufficient. However, because it is based on norms it is not necessary.

Because the integral in Eq. (3) is positive semi-definite the inequality can not be satisfied unless $\Re[\Lambda_1] < 0$. This condition is reminiscent of the discussion of stability of linear systems and Frozen coefficient analysis [25]. In addition, we have shown that the criteria for linear stability is $\Re[\Lambda_1] < 0$ up to second order in $\mathbf{x} - \mathbf{y}$ [3].

The stability criteria depends explicitly on the measure of the driving trajectory. Work by many authors indicates that the most unstable trajectories are likely to be those associated with fixed points of $\mathbf{F}$ and unstable periodic orbits of $\mathbf{F}$ with the shortest periods [26–29]. Given these observations it is believed that, if these trajectories are stable then the manifold $\mathbf{x} = \mathbf{y}$ is stable for all $\mathbf{x}$ [3].

Equation (3) has a geometrical interpretation that can be used to design couplings that yield stable synchronous motion. Both sides of Eq. (3) are functions of the elements of $\mathbf{DE}(\mathbf{0})$. Thus, $\left\langle \|\mathbf{P}^{-1}\left[\mathbf{DF}(\mathbf{x}) - \langle \mathbf{DF} \rangle\right]\mathbf{P}\| \right\rangle = \text{const.} \equiv C_1$ defines, $\Sigma_{\mathbf{x}}$, a family of surfaces in the parameter space of the elements of $\mathbf{DE}(\mathbf{0})$. In the same parameter space $-\Re[\Lambda_1] = \text{const.} \equiv C_2$ defines, $\Sigma_\Lambda$, a different family of surfaces. Therefore, the boundary of that portion of parameter space that yields linearly stable synchronous motion is the intersection of the family of surfaces $\Sigma_{\mathbf{x}}$ with the family of surfaces $\Sigma_\Lambda$. This approach is analytically shown in our manuscripts [3].

### 2. Anomalous scaling in non-normal transitions

Building on the work of [19], in [22] we consider the question of how non-normal bifurcations might be observed by developing scaling laws relating the size of finite amplitude perturbations ($\sigma_c$) which can drive a degenerate node nonlinearly unstable (i.e. we estimate the *subcritical threshold*). We also examined the effects of steady-state noise [23]. If $\epsilon$ is the bifurcation parameter (with $\epsilon = 0$ the linear threshold) then we are able to show that

$$\sigma_c \sim \epsilon^\gamma \tag{4}$$

with $\gamma = N - 1 + N/(n-1)$. Here $N$ is the number of degrees of freedom approaching threshold and $n$ is the order of the dominant nonlinearity. Fig. 1 shows that this scaling laws holds to a high degree of accuracy. The important point to notice is the exponential dependence on the number of degrees of freedom which are at threshold. Numerical studies have shown that if the exact degeneracy is weakly broken the scaling still holds. This suggests that non-normal transitions with only a handful of modes taking part can go unstable with little warning as the threshold is approached.

In most applications, of course, the dynamics on the center manifold is not isolated from the other degrees of freedom (which are damped but can be excited by external perturbations or noise). The hope is that these effects are weak and can be modeled stochastically. Such noise effects will cause fluctuations in the low-dimensional behavior.

To our knowledge there has been no work done on the noise response of such nonlinear systems (private communication, K. Lindenberg and G. Weiss). As might be guessed, non-normal systems are especially sensitive to noise. The most important such effect is *noise-driven escape* from the vicinity of the fixed point. A more familiar analogous behavior is escape from a metastable potential well. Although the fixed point is locally stable, the noise allows the system to explore the local region

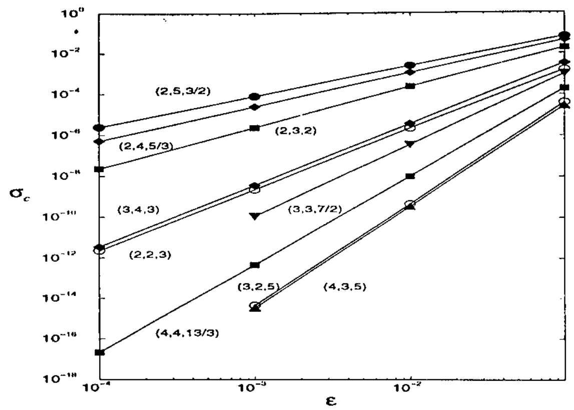


FIG. 1. Results of numerical tests of the scaling law 4. The notation is explained in the text.

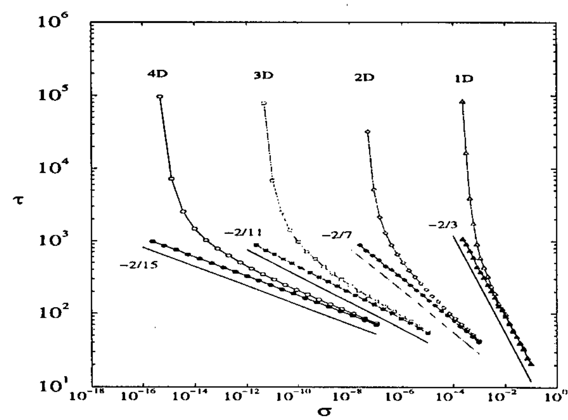


FIG. 2. Results of numerical tests of the dependence of the escape time, $\tau$ on both noise level, $\sigma$ and number of degenerate degrees of freedom, $N$.

and, if the noise is large enough, it can cross the separatrix boundary. For applications one should know how the average escape time, $\tau$, scales with various parameters such as the noise level, $\sigma$, and the number of degrees of freedom approaching threshold, $N$. This was explored numerically (see Fig. 2) and the results used to develop a scaling theory which explains all the important features [23].

### 3. Search for low-dimensional behavior in ABLE-ACE data

Recently, Jorgenson *et. al.* [30] claimed to find evidence of low-dimensional behavior in scintillation patterns associated with stellar observations. If substantiated this could lead to improved short-term predictability for adaptive optics applications. Efforts focussed on examining data sets supplied by Dr. Don Washburn of Phillips Lab taken during the ABLE-ACE campaign. This data was being subjected to a battery of tests designed to search for low-dimensional behavior in ABLE-ACE data. No evidence for such behavior was found.

## F. ACCOMPLISHMENTS AND NEW FINDINGS

The primary accomplishments of the grant period (4/1/97 - 12/31/97) are:

1. Observability/Detectability

- A rigorous sufficient criteria has been derived which guarantees linearly stable synchronization between dynamical systems when they are coupled in a drive–response manner. This result is an analytic method for constructing observers/detectors and for estimating the full state of a nonlinear plant from time series measurements [3].

2. Use of purely symbolic methods to extract correlation timescale estimates and to detect weak periodicities [11].

3. Anamolous scaling behavior in shear flow instabilities

   - An asymptotic estimate was developed for the threshold for nonlinear instability when the linearized dynamics is non-diagonalizable (i.e. strongly non-normal) [22]. This explains the anomalous threshold scaling observed numerically by Baggett and Trefethen.
   - Numerical studies were performed of noise-driven escape in non-normal systems. Relevant scaling laws and threshold estimates were extracted and explained [23].

4. We developed software to implement new diagnostic techniques which detect low-dimensional deterministic behavior. These diagnostics were applied to ABLE-ACE scintillation data. Our results suggest that, at least for the one data set examined, scintillation is fully 'turbulent' and probably not low-dimensional chaotic. Testing on at least a few more data sets will be necessary before a firm conclusions about low-dimensionality of scintillation data can be made.

## G. PERSONNEL SUPPORTED

**Senior Personnel**
Reggie Brown (1 month).
E. R. Tracy (1 month).

**Subcontractors**
John Wright (UCSD) (6 weeks).

**Post-doctoral fellow**
Xian-zhu Tang (1/1/97-8/1/97).

**Students**
Pablo Londero (W&M) (3 months).
D. J. Patil (UMd) (3 months).

**Programmer**
Dennis Weaver (3 weeks).

## H. PUBLICATIONS

1. "Designing coupling that guarantees synchronization between identical chaotic systems"; R. Brown and N. F. Rulkov, *Phys. Rev. Letts* **78**, 4189 (1997).
2. "Synchronization of chaotic systems: transverse stability of trajectories in invariant manifolds"; R. Brown and N. F. Rulkov, *Chaos* **7** 395 (1997).
3. "Parameter Uncertainties in Models of Equivariant Dynamical Systems"; R. Brown, V. In, and E. R. Tracy, *Physica* **102D**, 208 (1997).
4. "Symbol statistics and spatio-temporal systems"; X.–Z. Tang, E. R. Tracy, and R. Brown, *Physica* **102D** 253 (1997).
5. "Reconstruction of a Set of Equations from Scalar Time Series"; G. Gouesbet, L. Le Sceller, C. Letellier, and R. Brown, in *Annals of the New York Academy of Sciences, Vol 808: Nonlinear Signal and Image Anàlysis* eds. Buchler and Kandrup (New York Academy of Sciences, New York, 1997).
6. E. R. Tracy and X.-Z. Tang, "Anomalous scaling in Takens-Bogdanov bifurcations", to appear in Phys. Lett. A.
7. E. R. Tracy and X.-Z. Tang, "Takens-Bogdanov random walks", to appear in Phys. Rev. E.
8. X.-Z. Tang and E. R. Tracy, "Data compression and information retrieval via symbolization", to appear in Chaos.

## I. INTERACTIONS/TRANSITIONS

### 1. Presentations at meetings

1. Poster presentation (contributed) at Sherwood (Fusion Theory) Meeting, Madison WI (April, 1997), E. Tracy.
2. Invited Talk, Third SIAM Conference on Applications of Dynamical Systems, (May, 1997) Reggie Brown.
3. Contributed Poster, Third SIAM Conference on Applications of Dynamical Systems, (May, 1997), E. Tracy.
4. Seminar, Oak Ridge National Lab (October, 1997) E. Tracy.
5. Poster, 4th Experimental Chaos Conference, Boca Raton, FL (August, 1997) E. Tracy.

**1. Wm & Mary/AlliedSignal:**
**Performer:**
Professors E. R. Tracy and Reggie Brown, Dr. X.-Z. Tang & Ms. Sharon Burton.
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**Contact:**
Dr. R. Burne
Research Manager
(410)964-4159.
**Anticipated result:** Using the symbol statistics to detect transitions in complex systems, *e.g.* noise-driven turbulent flows.
**Application:** Early detection of rotating stall in turbines.

**2. Oak Ridge National Lab/Ford Motor Company:**
Work performed as part of a Cooperative Research and Development Agreement (CRADA), number ORNL-95-0337 titled "Engine Control Improvement Through Application of Chaotic Time Series Analysis".
**Performer:**
Dr. C. Stuart Daw
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Telephone: (615)574-0373.
**Customer:**
Ford Motor Co.
Dearborn, MI
**Contact:**
Dr. John Hoard
(313)594-1316.

**Anticipated result:** Using symbol statistics to do parameter fitting for an internal combustion engine model. (Aspects of the work are subject to a patent disclosure.)
**Application:** Improved feedback control for internal combustion engines to reduce NOx emission and increase fuel efficiency.

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[11] X.-Z. Tang and E. R. Tracy, "Data compression and information retrieval using symbolization", to appear in Chaos.

[12] Many of these results can be found in one of the many review papers that have appeared. For example, see the reviews by Abarbanel, H. D. I., R. Brown, J. J. Sidorowich and L. S. Tsimring, "The analysis of observed chaotic data in physical systems", *Rev. Mod. Phys.* **65**, 1331 (1993); M. Casdagli, D. Des Jardins, S. Eubank, J. D. Farmer, J. Gibson, N. Hunter, and J. Theiler, "Nonlinear modeling of chaotic time series: theory and applications", in *Applied Chaos*, ed. J. Kim (Addison-Wesley, Reading, MA) (1992). In addition new methods appear each month in the journals.

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